



Some Solutions for the Large Deflections of Uniformly Loaded Circular Membranes

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Summary

Inconsistent citations in the literature and questions about convergence prompt reexamination of Hencky's classic solution for the large deflections of a clamped, circular isotropic membrane under uniform pressure. This classic solution is observed actually to be for uniform lateral loading because the radial component of the pressure acting on the deformed membrane is neglected. An algebraic error in Hencky's solution is corrected, additional terms are retained in the power series to assess convergence, and results are obtained for two additional values of Poisson's ratio.

To evaluate the importance of the neglected radial component of the applied pressure, the problem is reformulated with this component included and is solved, with escalating algebraic complexity, by a similar power-series approach. The two solutions agree quite closely for lightly loaded membranes and diverge slowly as the load intensifies. Differences in maximum stresses and deflections are substantial only when stresses are very high. The more nearly spherical deflection shape of the membrane under true pressure loading suggests that a near-parabolic membrane reflector designed on the basis of the more convenient Hencky theory would not perform as well as expected.

In addition, both theories are found to yield closed-form, nonuniform membrane-thickness distributions that produce parabolic middle-surface deflections under loading. Both distributions require that the circular boundary expand radially in amounts that depend on material and loading parameters.

Introduction

Concepts for orbiting inflatable reflectors are of interest primarily because of their relative mechanical simplicity, high area-to-mass ratio, and compact packaging characteristics. Essential to the design and fabrication of inflatable reflectors is the ability to predict the reflector shape upon inflation. When neither a deep reflector nor extreme surface precision is required, an attractively simple configuration is an initially flat and unstressed circular elastic membrane attached at its perimeter to a stiff ring and subjected to differential pressure. While uniform loading applied to a constant-thickness membrane will not produce the exact paraboloidal reflector shape that is desired in numerous applications, the shape difference may be small enough for many purposes.

This paper, because of inconsistent citations in the literature (e.g., ref. 1) and questions about convergence, reexamines Hencky's original analysis (ref. 2) of the large deflections of a clamped, circular membrane under uniform pressure. Hencky's power-series approach is again employed, an algebraic error is corrected, more terms are retained to assess convergence, and results are generated for two additional values of Poisson's ratio.

Also, because Hencky's problem actually involves uniform *lateral* loading (i.e., the radial component of pressure on the deformed membrane is neglected), the boundary value problem for true uniform-pressure loading is formulated and solved. Results for lateral deflections and membrane stresses from both Hencky's solution (corrected) and the uniform-pressure solution are presented in tabular and graphical form, and comparisons are made between the two solutions.

In addition, although neither problem solution yields an exactly paraboloidal deflection shape, nonuniform membrane-thickness distributions that yield such shapes can be found for both loading conditions. These distributions are derived in the appendix.

Symbols

a	radius of membrane
$a_{2n}, b_{2n}, n_{2m}, w_{2m}$	coefficients in power series
E	modulus of elasticity
h	thickness of membrane

N	dimensionless meridional stress resultant ($N_r/(Eh)$ in Hencky's problem, N_r/pa in uniform-pressure problem)
N_r, N_θ	meridional and circumferential stress resultants, respectively
p	uniform loading (lateral in Hencky's problem, normal in uniform-pressure problem)
q	dimensionless loading parameter, pa/Eh
r, θ	plane polar coordinates
u	radial displacement
W	dimensionless lateral deflection, w/a
w	lateral deflection
α	constant, $(3 + \mu)/2$
$\epsilon_r, \epsilon_\theta$	meridional and circumferential extensional strains, respectively
μ	Poisson's ratio
ρ	dimensionless radial coordinate, r/a

Analysis

The problem of the large deflections of a uniform-thickness, circular isotropic elastic membrane, clamped at its boundary without pre-tension and subjected to uniform lateral loading (Hencky's problem), is addressed first. Next, the same configuration under uniform pressure is analyzed.

Uniform Lateral Loading (Hencky's Problem)

The governing equations are

radial equilibrium

$$N_\theta = \frac{d}{dr}(rN_r) \quad (1)$$

lateral equilibrium

$$N_r \frac{dw}{dr} = -\frac{pr}{2} \quad (2)$$

where N_r and N_θ are, respectively, meridional and circumferential stress resultants, r is the radial coordinate, w is the lateral deflection, and p is uniform lateral loading;

stress-strain

$$N_\theta - \mu N_r = Eh\epsilon_\theta \quad (3)$$

$$N_r - \mu N_\theta = Eh\epsilon_r \quad (4)$$

strain-displacement

$$\epsilon_\theta = \frac{u}{r} \quad (5)$$

$$\epsilon_r = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 \quad (6)$$

where u is the radial displacement and μ is Poisson's ratio. Equation (2) is the result of one integration of the original lateral equilibrium equation (ref. 2) and use of the symmetry condition $dw/dr(0) = 0$ along with the regularity of N_r at $r = 0$. The boundary conditions at the clamped edge are

$$w(a) = 0 \quad (7)$$

$$u(a) = 0 \quad (8)$$

Combining equations (1) and (3) through (6), and defining dimensionless quantities $W = w/a$, $N = N_r/(Eh)$, $\rho = r/a$, and $q = pa/(Eh)$, the resulting equations are

$$\rho \frac{d}{d\rho} \left[\frac{d}{d\rho} (\rho N) + N \right] + \frac{1}{2} \left(\frac{dW}{d\rho} \right)^2 = 0 \quad (9)$$

$$N \frac{dW}{d\rho} = -\frac{1}{2} q \rho \quad (10)$$

Substitution of equation (10) into equation (9) gives

$$N^2 \frac{d}{d\rho} \left[\frac{d}{d\rho} (\rho N) + N \right] + \frac{1}{8} q^2 \rho = 0 \quad (11)$$

After Hencky's approach, the following forms for $N(\rho)$ and $W(\rho)$ in equations (10) and (11) are assumed:

$$N(\rho) = \frac{1}{4} q^{2/3} \sum_0^\infty b_{2n} \rho^{2n} \quad (12a)$$

$$W(\rho) = q^{1/3} \sum_0^\infty a_{2n} (1 - \rho^{2n+2}) \quad (12b)$$

By these choices, equations (10) and (11) become independent of q , so that the power series in equations (12a) and (12b), once determined for a specified value of μ , are valid for all q . Also, substituting equation (12a) into the radial equilibrium equation gives, for the dimensionless circumferential stress resultant,

$$\frac{N_\theta}{Eh} = \frac{1}{4} q^{2/3} \sum_0^\infty (2n+1) b_{2n} \rho^{2n}$$

Note that the form assumed for $W(\rho)$ ensures term-by-term satisfaction of the boundary condition on w , equation (7). Then equation (11) becomes

$$\left\{ b_0 + b_2 \rho^2 + b_4 \rho^4 + b_6 \rho^6 + \dots \right\}^2 \left\{ 4(2)b_2 \rho + 6(4)b_4 \rho^3 + 8(6)b_6 \rho^5 + 10(8)b_8 \rho^7 + \dots \right\} = -8\rho \quad (13)$$

Expanding the left side of equation (13) and equating coefficients of like powers of p yields the following relations between $b_0, b_2, b_4, b_6, \dots$, which can be solved successively for b_2, b_4, b_6, \dots in terms of b_0 :

$$b_0^2 b_2 = -1 \quad (14)$$

$$3b_0^2 b_4 + 2b_0 b_2^2 = 0 \quad (15)$$

$$6b_0^2 b_6 + 6b_0 b_2 b_4 + b_2(b_2^2 + 2b_0 b_4) = 0 \quad (16)$$

$$10b_0^2 b_8 + 12b_0 b_2 b_6 + 3b_4(b_2^2 + 2b_0 b_4) + b_2(2b_0 b_6 + 2b_2 b_4) = 0 \quad (17)$$

$$15b_0^2 b_{10} + 20b_0 b_2 b_8 + 6b_6(b_2^2 + 2b_0 b_4) + 3b_4(2b_0 b_6 + 2b_2 b_4) + b_2(b_4^2 + 2b_0 b_8 + 2b_2 b_6) = 0 \quad (18)$$

$$21b_0^2 b_{12} + 30b_0 b_2 b_{10} + 10b_8(b_2^2 + 2b_0 b_4) + 6b_6(2b_0 b_6 + 2b_2 b_4) + 3b_4(b_4^2 + 2b_0 b_8 + 2b_2 b_6) + b_2(2b_0 b_{10} + 2b_2 b_8 + 2b_4 b_6) = 0 \quad (19)$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

In all, the first 10 of these equations have been derived and solved. The results are

$$b_2 = -\frac{1}{b_0^2} \quad (20)$$

$$b_4 = -\frac{2}{3b_0^5} \quad (21)$$

$$b_6 = -\frac{13}{18b_0^8} \quad (22)$$

$$b_8 = -\frac{17}{18b_0^{11}} \quad (23)$$

$$b_{10} = -\frac{37}{27b_0^{14}} \quad (24)$$

$${}^1b_{12} = -\frac{1205}{567b_0^{17}} \quad (25)$$

$$b_{14} = -\frac{219241}{63504b_0^{20}} \quad (26)$$

$$b_{16} = -\frac{6634069}{1143072b_0^{23}} \quad (27)$$

$$b_{18} = -\frac{51523763}{5143824b_0^{26}} \quad (28)$$

$$b_{20} = -\frac{998796305}{56582064b_0^{29}} \quad (29)$$

In obtaining equations (14) through (29), extensive use has been made of the symbolic calculation capabilities of MathematicaTM software (ref. 3). The lead coefficient b_0 is evaluated through satisfaction of the remaining boundary condition, equation (8), which in dimensionless form is

$$\left\{ \rho \left[\frac{d}{d\rho} (\rho N) - \mu N \right] \right\}_{\rho=1} = 0$$

or, equivalently,

$$(1 - \mu)b_0 + (3 - \mu)b_2 + (5 - \mu)b_4 + (7 - \mu)b_6 + \dots = 0 \quad (30)$$

Substituting from equations (20) through (29) into equation (30) yields an equation in b_0 :

$$\begin{aligned} & (1 - \mu)b_0 - (3 - \mu)\frac{1}{b_0^2} - (5 - \mu)\frac{2}{3b_0^5} - (7 - \mu)\frac{13}{18b_0^8} - (9 - \mu)\frac{17}{18b_0^{11}} \\ & - (11 - \mu)\frac{37}{27b_0^{14}} - (13 - \mu)\frac{1205}{567b_0^{17}} - (15 - \mu)\frac{219241}{63504b_0^{20}} \\ & - (17 - \mu)\frac{6634069}{1143072b_0^{23}} - (19 - \mu)\frac{51523763}{5143824b_0^{26}} - (21 - \mu)\frac{998796305}{56582064b_0^{29}} + \dots = 0 \end{aligned} \quad (31)$$

¹In reference 2, Hencky terminated this sequence with b_{12} , for which he reported the erroneous value of $-407/189b_0^{17}$. Hencky's error was corrected by J. D. Campbell in reference 4, where Hencky's problem also was generalized to include arbitrary initial tension.

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For each specified value of μ , solution of a suitably truncated version of equation (31) yields the single value of b_0 for which $b_0 > 1$. (The infinite series in eq. (31) diverges for $-1 \leq b_0 \leq 1$, and $b_0 \leq -1$ implies compressive stresses.) To investigate the convergence of b_0 with the number of terms retained, a sequence of truncated versions of equation (31) containing from 2 to 11 terms can be solved to provide a sequence of values of b_0 . This process has been done for each of the three values of μ considered herein. The results with 11 terms retained, which differ very little from those with only 6 terms retained (as was done in ref. 2), are presented in the following table.

μ	b_0
0.2	1.6827
0.3	1.7244 ^a
0.4	1.7769

^aLargely because of an error in the evaluation of b_{12} , Hencky obtained $b_0 = 1.713$. By apparently truncating equation (12a) of the present paper after the b_{12} term, Campbell obtained a value for b_0 of 1.724, which is very close to the value of 1.7244 found herein.

With the value of b_0 in hand, the truncated power series for both stress resultants and the radial displacement are completely determined. To obtain the coefficients in the series for $W(\rho)$, expressions (12a) and (12b) are inserted into equation (10), which becomes

$$\left\{ b_0 + b_2 \rho^2 + b_4 \rho^4 + b_6 \rho^6 + \dots \right\} \left\{ a_0 + 2a_2 \rho^2 + 3a_4 \rho^4 + 4a_6 \rho^6 + \dots \right\} = 1 \quad (32)$$

Equating coefficients of like powers of ρ yields the following relations between the two sets of coefficients:

$$\left. \begin{aligned} b_0 a_0 &= 1 \\ 2b_0 a_2 + b_2 a_0 &= 0 \\ 3b_0 a_4 + 2b_2 a_2 + b_4 a_0 &= 0 \\ 4b_0 a_6 + 3b_2 a_4 + 2b_4 a_2 + b_6 a_0 &= 0 \\ 5b_0 a_8 + 4b_2 a_6 + 3b_4 a_4 + 2b_6 a_2 + b_8 a_0 &= 0 \\ \vdots & \\ \vdots & \end{aligned} \right\} \quad (33)$$

Successive solution of equations (33) for a_0, a_2, a_4, \dots combined with equations (20) through (29) gives

$$\left. \begin{aligned} a_0 &= \frac{1}{b_0} \\ a_2 &= \frac{1}{2b_0^4} \\ a_4 &= \frac{5}{9b_0^7} \\ a_6 &= \frac{55}{72b_0^{10}} \\ a_8 &= \frac{7}{6b_0^{13}} \\ a_{10} &= \frac{205}{108b_0^{16}} \\ a_{12} &= \frac{17051}{5292b_0^{19}} \\ a_{14} &= \frac{2864485}{508032b_0^{22}} \\ a_{16} &= \frac{103863265}{10287648b_0^{25}} \\ a_{18} &= \frac{27047983}{1469664b_0^{28}} \\ a_{20} &= \frac{42367613873}{1244805408b_0^{31}} \end{aligned} \right\} \quad (34)$$

Substitution of these coefficients into equation (12b) gives the dimensionless lateral displacement $W(\rho)$. At this point, truncated power series can be written for all stress resultants and displacements. Tables and graphs of stress resultants and displacements are deferred until comparisons can be made with corresponding results from the solution for uniform pressure.

Uniform Pressure

All of the equations governing the uniform-pressure problem are identical to those of Hencky's problem (i.e., eqs. (1) through (8)) except for equation (1), the radial equilibrium equation. The new radial equilibrium equation can be obtained by application of the principle of virtual work or by summing forces on an infinitesimal element of the deformed membrane. By noting that wherever the

membrane has nonzero slope the normal pressure has a radial component (neglected in Hencky's problem), the radial equilibrium equation is seen to be

$$N_{\theta} = \frac{d}{dr}(rN_r) - pr \frac{dw}{dr} \quad (35)$$

Combining equations (35) and (3) through (6) yields

$$N_r^2 \left(r^2 \frac{d^2 N_r}{dr^2} + 3r \frac{dN_r}{dr} \right) - \frac{p^2 r^3}{2} \frac{dN_r}{dr} + \left(\frac{3+\mu}{2} \right) p^2 r^2 N_r + \frac{Eh p^2 r^2}{8} = 0 \quad (36)$$

which must be satisfied along with

$$N_r \frac{dw}{dr} = -\frac{pr}{2} \quad (37)$$

The boundary condition on radial displacement takes the slightly more involved form

$$\left\{ r \left[\frac{d}{dr}(rN_r) - \mu N_r - pr \frac{dw}{dr} \right] \right\}_{r=a} = 0 \quad (38)$$

The nondimensionalization employed in solving Hencky's problem does not lead here to versions of equations (36) and (37) that are independent of q . Nevertheless, the nondimensionalization was used initially in the solution process, but was abandoned when inaccuracies were encountered in the numerical evaluation of series coefficients. With the definitions $\rho = r/a$, $W = w/a$, $N = N_r/pa$, and $q = (pa)/(Eh)$, equations (36) and (37) become

$$N^2 \left(\rho^2 \frac{d^2 N}{d\rho^2} + 3\rho \frac{dN}{d\rho} \right) - \frac{1}{2} \rho^3 \frac{dN}{d\rho} + \alpha \rho^2 N + \frac{1}{8} \frac{\rho^2}{q} = 0 \quad (q > 0) \quad (39)$$

where $\alpha = (3 + \mu)/2$, and

$$N \frac{dW}{d\rho} = -\frac{1}{2} \rho \quad (40)$$

Substituting the assumed form

$$N(\rho) = \sum_{m=0}^{\infty} n_{2m} \rho^{2m} \quad (41)$$

into equation (39) and equating coefficients of like powers of p gives the following sequence of relations between $n_0, n_2, n_4, n_6, \dots$, which can be solved for n_2, n_4, n_6, \dots in terms of n_0 :

$$\left. \begin{aligned} 8n_0^2n_2 + \alpha n_0 + \frac{1}{8q} &= 0 \\ 24n_0^2n_4 + 16n_0n_2^2 + (\alpha - 1)n_2 &= 0 \\ 48n_0^2n_6 + 48n_0n_2n_4 + 8(2n_0n_2 + n_2^2)n_2 + (\alpha - 2)n_4 &= 0 \\ 80n_0^2n_8 + 96n_0n_2n_6 + 24(2n_0n_4 + n_2^2)n_4 + 16(n_0n_6 + n_2n_4)n_2 + (\alpha - 3)n_6 &= 0 \\ 120n_0^2n_{10} + 160n_0n_2n_8 + 48(2n_0n_4 + n_2^2)n_6 + 48(n_0n_6 + n_2n_4)n_4 \\ + 8(2n_0n_8 + 2n_2n_6 + n_4^2)n_2 + (\alpha - 4)n_8 &= 0 \\ 168n_0^2n_{12} + 240n_0n_2n_{10} + 80(2n_0n_4 + n_2^2)n_8 + 96(n_0n_6 + n_2n_4)n_6 \\ + 24(2n_0n_8 + 2n_2n_6 + n_4^2)n_4 + 16(n_0n_{10} + n_2n_8 + n_4n_6)n_2 + (\alpha - 5)n_{10} &= 0 \\ \vdots & \end{aligned} \right\} \quad (42)$$

The first eight of these equations have been derived and solved, although only the first six are shown in equations (42) because of rapidly increasing algebraic complexity. For the same reason, only the first four solutions are shown here.

$$\left. \begin{aligned} n_2 &= -\frac{(1 + 8\alpha n_0 q)}{64n_0^2 q} \\ n_4 &= -\frac{(1 + 8\alpha n_0 q)(1 + 4n_0 q + 4\alpha n_0 q)}{6144n_0^5 q^2} \\ n_6 &= -\frac{(1 + 8\alpha n_0 q)(13 + 96n_0 q + 128\alpha n_0 q + 128n_0^2 q^2 + 576\alpha n_0^2 + 256\alpha^2 n_0^2 q^2)}{4718592n_0^8 q^3} \\ n_8 &= -(1 + 8\alpha n_0 q)(39 + 366n_0 q + 670\alpha n_0 q + 288n_0^3 q + 960n_0^2 q^2 + 4608\alpha n_0^2 q^2 \\ &\quad + 3584\alpha^2 n_0^2 q^2 + 4608n_0^4 q^2 + 768n_0^3 q^3 + 6272\alpha n_0^3 q^3 + 14208\alpha^2 n_0^3 q^3 \\ &\quad + 5632\alpha^3 n_0^3 q^3 + 36864\alpha n_0^5 q^3 - 18432\alpha^2 n_0^5 q^3)/754974720n_0^{11} q^4 \\ \vdots & \end{aligned} \right\} \quad (43)$$

In the development of equations (42) and (43), the Mathematica system was again used extensively. Substitution of equations (43) into equation (41) yields the power series for $N(p)$ in terms of the yet-undetermined coefficient n_0 that, in this problem, depends on both μ and q . However, before the

remaining boundary condition, equation (38), can be applied to evaluate n_0 , equation (40) must be used to express the coefficients in the assumed series

$$W(\rho) = \sum_0^{\infty} w_{2n}(1 - \rho^{2n+2}) \quad (44)$$

in terms of n_0 . Substituting equations (41) and (44) into equation (40) and equating coefficients of like powers of ρ gives the system of simultaneous equations

$$\left. \begin{aligned} n_0 w_0 &= \frac{1}{4} \\ 2n_0 w_2 + n_2 w_0 &= 0 \\ 3n_0 w_4 + 2n_2 w_2 + n_4 w_0 &= 0 \\ 4n_0 w_6 + 3n_2 w_4 + 2n_4 w_2 + n_6 w_0 &= 0 \\ 5n_0 w_8 + 4n_2 w_6 + 3n_4 w_4 + 2n_6 w_2 + n_8 w_0 &= 0 \\ \vdots & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \end{aligned} \right\} \quad (45)$$

from which w_0, w_2, w_4, \dots can be found in terms of n_0 . For the sake of brevity, only the first five solutions are shown:

$$\left. \begin{aligned} w_0 &= 1/4n_0 \\ w_2 &= (1 + 8\alpha n_0 q)/512n_0^4 q \\ w_4 &= (1 + 8\alpha n_0 q)(5 + 8n_0 q + 32\alpha n_0 q)/147456n_0^7 q^2 \\ w_6 &= (1 + 8\alpha n_0 q)(55 + 192n_0 q + 704\alpha n_0 q + 128n_0^2 q^2 + 1344\alpha n_0^2 q^2 \\ &\quad + 2176\alpha^2 n_0^2 q^2)/75497472n_0^{10} q^3 \\ w_8 &= (1 + 8\alpha n_0 q)(259 + 1366n_0 q + 5030\alpha n_0 q + 288n_0^3 q + 1920n_0^2 q^2 \\ &\quad + 19008\alpha n_0^2 q^2 + 31744\alpha^2 n_0^2 q^2 + 4608n_0^4 q^2 + 768n_0^3 q^3 \\ &\quad + 13952\alpha n_0^3 q^3 + 65408\alpha^2 n_0^3 q^3 + 64512\alpha^3 n_0^3 q^3 + 36864\alpha n_0^5 q^3 \\ &\quad - 18432\alpha^2 n_0^5 q^3)/15099494400n_0^{13} q^4 \\ \vdots & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \end{aligned} \right\} \quad (46)$$

Now n_0 can be evaluated through satisfaction of the boundary condition on radial displacement, equation (38), which has the dimensionless form

$$\left\{ \rho \left[\frac{d}{d\rho}(\rho N) - \mu N - \rho \frac{dW}{d\rho} \right] \right\}_{\rho=1} = 0 \quad (47)$$

Substitution of equations (41) and (44) into equation (47) yields

$$\left(\frac{1-\mu}{2}\right)n_0 + \left(\frac{3-\mu}{2}\right)n_2 + \left(\frac{5-\mu}{2}\right)n_4 + \left(\frac{7-\mu}{2}\right)n_6 + \dots + w_0 + 2w_2 + 3w_4 + 4w_6 + \dots = 0 \quad (48)$$

in which w_0, w_2, w_4, \dots are given in terms of n_0 by equations (46).

Substitution of equations (43), giving n_2, n_4, n_6, \dots in terms of n_0 , and equations (46), giving w_0, w_2, w_4, \dots in terms of n_0 , into equation (48) yields an equation for the lead coefficient n_0 . After specification of values for μ and q , various truncations of this equation can be solved until a satisfactorily converged value of n_0 is obtained. This value of n_0 can then be used along with equations (43) to write the explicit truncated series for $N(\rho)$. Similarly, equations (46) can be inserted into equation (44) to produce the series for $W(\rho)$. With these two series in hand, expressions for all other variables of interest can be generated.

Results and Discussion

Lateral Deflections

The variables of greatest interest in the design of membrane reflectors are the lateral deflection w and the meridional stress resultant N_r . In figure 1, the dimensionless lateral deflection $W = w/a$ for Hencky's problem is plotted as a function of $\rho = r/a$, the dimensionless radial coordinate, for $q = 0.001, 0.01$, and 0.1 and $\mu = 0.2, 0.3$, and 0.4 .

Note that, in Hencky's problem, the family of curves for a fixed value of q can be used to produce curves for any other value of q by appropriate adjustment of the vertical scale. The same is not true, however, in the solution to the uniform-pressure problem. Figure 2 contains plots of the lateral deflection from the uniform-pressure problem for the same set of values of μ and q shown in figure 1. In both solutions, as deflections increase with q , so do the Poisson's-ratio-related differences among them, with larger values of μ corresponding to smaller deflections.

To illustrate the effect on lateral deflection of retaining the radial component of the pressure, results from figures 1 and 2, for $\mu = 0.3$ only, are plotted together in figure 3 for $q = 0.001, 0.01$, and 0.1 . For $q = 0.001$, the deflections for the two loading cases are nearly identical, as would be expected, because the radial component of the pressure is proportional to dw/dr , which is everywhere small when q is small. In fact, even for $q = 0.01$, the deflection differences are still very small although they could have some significance for application to high-precision surfaces for electromagnetic reflectors. For $q = 0.1$, a value that corresponds to very high membrane stresses, a basic difference between the deflection shapes is quite apparent. The uniform-pressure loading causes a more nearly spherical shape, one that is even farther removed from the ideal paraboloid than the shape predicted by the solution to Hencky's problem. Thus, the present results indicate that a reflector design based on the solution to Hencky's problem would likely overestimate electromagnetic performance. With regard to achieving the ideal deflection shape, nonuniform membrane-thickness distributions that lead to paraboloidal middle-surface deflection shapes can be found in both problems. Details of the analyses are presented in the appendix.

Another useful way of viewing the results is by envisioning the loaded membrane as a nonlinear spring. Figure 4 contains plots of the dimensionless loading parameter q as a function of the dimensionless center deflection $w(0)/a$ from each solution for the three values of Poisson's ratio. Both solutions depict systems that are stiffening with increasing load, as evidenced by the increasing slopes, especially at the larger values of q , with the normal pressure solution representing the stiffer system.

Stress Resultants

Figure 5 contains plots of the dimensionless meridional stress resultant $N_r/(Eh)$ for $q = 0.001, 0.01$, and 0.1 , with $\mu = 0.2, 0.3$, and 0.4 . For $q = 0.001$, the curves for Hencky's problem and for the normal-pressure case are indistinguishable on the figure. For $q = 0.01$, differences between the two solutions still appear to be quite small, particularly at the scale used for the figure. The differences become more pronounced as q continues to increase until, for $q = 0.1$, the uniform-pressure stresses substantially exceed those of Hencky's problem at the center, while the opposite is true at the boundary. The following table contains a list of values of the maximum dimensionless stress resultant $N_r(0)/(Eh)$ from both solutions as a function of q for all three values of μ .

q	Uniform pressure			Hencky		
	$\mu = 0.2$	$\mu = 0.3$	$\mu = 0.4$	$\mu = 0.2$	$\mu = 0.3$	$\mu = 0.4$
0.001	0.00425	0.00435	0.00448	0.00421	0.00431	0.00444
.002	.00677	.00693	.00714	.00668	.00684	.00705
.003	.00890	.00911	.00938	.00875	.00897	.00924
.005	.0126	.0129	.0133	.0123	.0126	.0130
.008	.0173	.0177	.0183	.0168	.0172	.0178
.010	.0202	.0207	.0213	.0195	.0200	.0206
.020	.0326	.0333	.0343	.0310	.0318	.0327
.030	.0433	.0443	.0456	.0406	.0416	.0429
.050	.0622	.0637	.0655	.0571	.0585	.0603
.100	.1032	.1057	.1086	.0906	.0929	.0957

Thus, while it appears that for all $q > 0$ the solution to Hencky's problem understates the maximum membrane stress resultant ($N_r(0)$, which is equal to $N_\theta(0)$ in either solution), the discrepancy is substantial only for large q , in which case the stress resultants predicted by either theory are very large.

Concluding Remarks

The stresses and large deflections of a clamped, circular membrane have been analyzed for two types of loading: uniform lateral loading (Hencky's problem) and uniform-pressure loading. Results for the two types of loading have been shown to agree quite closely when the membrane is lightly loaded and to diverge slowly as load intensity is increased. The deflection shapes from the uniform-pressure solution are more nearly spherical than those from the solution to Hencky's problem, especially at the higher load intensities. This result suggests that a near-paraboloidal inflatable reflector designed on the basis of the more convenient Hencky solution would overestimate electromagnetic performance because of shape-accuracy shortcomings.

Both problems were found to yield closed-form nonuniform membrane-thickness distributions that produce exact paraboloidal middle-surface deflection shapes. The fabrication of such variable-thickness membranes appears to pose a daunting challenge.

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Appendix

Membrane-Thickness Distributions That Yield Parabolic Deflection Shapes

To find thickness distributions that yield parabolic middle-surface deflections under uniform loading, the governing equations are approached under the assumptions that $w(r)$ is a prescribed parabola and the thickness $h(r)$ is to be determined. The variable-thickness version of Hencky's problem is addressed first, followed by the uniform-pressure problem.

Hencky's Problem

The governing equations are repeated here for convenient reference.

$$N_\theta = \frac{d}{dr}(rN_r) \quad (\text{A1})$$

$$N_r \frac{dw}{dr} = -\frac{pr}{2} \quad (\text{A2})$$

$$Eh\varepsilon_r = N_r - \mu N_\theta \quad (\text{A3})$$

$$Eh\varepsilon_\theta = N_\theta - \mu N_r \quad (\text{A4})$$

$$\varepsilon_r = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 \quad (\text{A5})$$

$$\varepsilon_\theta = \frac{u}{r} \quad (\text{A6})$$

A parabolic deflection shape is assumed.

$$w(r) = w_0 + W\left(1 - \frac{r^2}{a^2}\right) \quad (\text{A7})$$

where w_0 is an unspecified rigid-body translation and W is the strain-related maximum deflection, so that

$$\frac{dw}{dr} = -\frac{2Wr}{a^2} \quad (\text{A8})$$

Then equations (A1) and (A2) yield

$$N_r = N_\theta = \frac{pa^2}{4W} \quad (\text{A9})$$

Combining this result with equations (A3) through (A6) gives

$$Eh\left(\frac{du}{dr} + 2\frac{W^2 r^2}{a^4}\right) = (1 - \mu)\frac{pa^2}{4W} \quad (\text{A10})$$

and

$$u = \frac{(1 - \mu)pa^2 r}{4WEh} \quad (\text{A11})$$

Substitution of equation (A11) into (A10) and some simplification yield

$$\frac{d}{dr}\left(\frac{1}{h}\right) = -\frac{8EW^3r}{(1-\mu)pa^6} \quad (\text{A12})$$

After integration, with the definition $h(0) = h_0$, the solution is found to be

$$h(r) = \frac{h_0}{\left[1 - \frac{4Eh_0W^3r^2}{(1-\mu)pa^6}\right]} \quad (\text{A13})$$

which, on physical grounds, requires satisfaction of the inequality

$$\frac{4Eh_0W^3}{(1-\mu)pa^4} < 1 \quad (\text{A14})$$

Note that, for sufficiently small values of the quotient in the denominator of equation (A14), the radial thickness variation is very nearly parabolic. The focal length F of the parabolic deflection shape is given by $F = a^2/4W$, so that $W = a/8(F/D)$, where $D = 2a$ is the diameter of the circular reflector. Then, in terms of dimensionless ratios, an alternate form for equation (A13) is

$$h(r) = \frac{h_0}{\left[1 - \frac{1}{128(1-\mu)} \frac{E}{p} \frac{h_0}{a} \frac{(r/a)^2}{(F/D)^3}\right]} \quad (\text{A15})$$

Also, equation (A9) can be rewritten as

$$N_r = N_\theta = pF = \sigma_0 h_0 \quad (\text{A16})$$

in which σ_0 is the (maximum) membrane stress at $r = 0$, where $h(r)$ is minimum. Thus, with the focal length F prescribed and an acceptable working value chosen for σ_0 , equation (A16) enables a family of choices for p and h_0 that, along with values for D and material properties E and μ , can be inserted into equation (A15) to produce a family of candidate thickness profiles from which to choose. Clearly, the larger the value of F/D (i.e., the shallower the reflector) the greater the freedom in assigning values to the remaining parameters.

Note that this solution cannot be made consistent with the original boundary condition $u(a) = 0$. From equations (A4), (A6), and (A9), the radial displacement takes the form

$$u(r) = \frac{1-\mu}{E} \frac{pa^2r}{4Wh(r)} \quad (\text{A17})$$

from which it is clear that $u(r)$ must be positive for all $r > 0$. Thus, in the variable-thickness version of Hencky's problem, achieving the parabolic deflection shape would also require an appropriate uniform expansion of the circular boundary.

Uniform-Pressure Problem

The governing equations for this problem are equations (A2) through (A6) along with

$$\frac{d}{dr}(rN_r) - N_\theta = pr \frac{dw}{dr} \quad (\text{A18})$$

in place of equation (A1). Assumption of the same parabolic displacement function, equation (A7), leads to

$$N_r = \frac{pa^2}{4W} \quad (\text{A19})$$

and

$$N_\theta = \frac{pa^2}{4W} + 2pW \frac{r^2}{a^2} \quad (\text{A20})$$

Insertion of equations (A19) and (A20) into equations (A3) and (A4) followed by substitution of the results into equations (A5) and (A6) leads to

$$Eh \left(\frac{du}{dr} + 2 \frac{W^2 r^2}{a^4} \right) = (1 - \mu) \frac{pa^2}{4W} - 2\mu pW \frac{r^2}{a^2} \quad (\text{A21})$$

$$Eh \frac{u}{r} = (1 - \mu) \frac{pa^2}{4W} + 2pW \frac{r^2}{a^2} \quad (\text{A22})$$

Eliminating $u(r)$ between equations (A21) and (A22) gives

$$\frac{d}{dr} \left\{ \frac{1}{h} \left[(1 - \mu) \frac{pa^2 r}{4W} + 2 \frac{pW r^3}{a^2} \right] \right\} = \frac{1}{h} \left[(1 - \mu) \frac{pa^2}{4W} - 2 \frac{\mu pW r^2}{a^2} \right] - 2 \frac{EW^2 r^2}{a^4} \quad (\text{A23})$$

Differentiation followed by simplification, where possible, leads to a linear first-order differential equation in $1/h(r)$ with variable coefficients. Its solution with the help of an integrating factor yields

$$h(r) = h_0 \frac{\left[1 + \frac{8W^2 r^2}{(1 - \mu)a^4} \right]^{(3 + \mu)/2}}{1 - \frac{EWh_0}{(3 + \mu)pa^2} \left\{ \left[1 + \frac{8W^2 r^2}{(1 - \mu)a^4} \right]^{(3 + \mu)/2} - 1 \right\}} \quad (\text{A24})$$

where, again, $h_0 = h(0)$. Clearly, $h(r)$ has its minimum value at $r = 0$.

An alternate form of equation (A24) in terms of the dimensionless ratios employed earlier is

$$h(r) = h_0 \frac{\left[1 + \frac{(r/a)^2}{8(1 - \mu)(F/D)^2} \right]^{(3 + \mu)/2}}{1 - \frac{1}{8(3 + \mu)} \frac{E}{p} \frac{h_0}{a} \frac{1}{(F/D)^2} \left\{ \left[1 + \frac{(r/a)^2}{8(1 - \mu)(F/D)^2} \right]^{(3 + \mu)/2} - 1 \right\}} \quad (\text{A25})$$

This equation coupled with

$$N_r(0) = N_\theta(0) = \frac{pa^2}{4W}$$

or, equivalently,

$$pF = \sigma_0 h_0 \tag{A26}$$

can be used along with combinations of material and geometric properties, some of which are likely to be mandated by a particular application, to produce various candidate thickness distributions from which to choose. As in the preceding problem, achieving the parabolic deflection shape would require appropriate expansion of the circular boundary, which is apparent from equation (A22).

The fabrication of a membrane with a thickness distribution mandated by either of these theories may be beyond the current state of film manufacturing technology. Of the two families of thickness distributions, the one appropriate to Hencky's problem may be nearer to achievable, possibly by use of a rotating, ultraflat film-fabrication table, because it is very nearly parabolic when the thickness variation relative to the minimum thickness is not too great.

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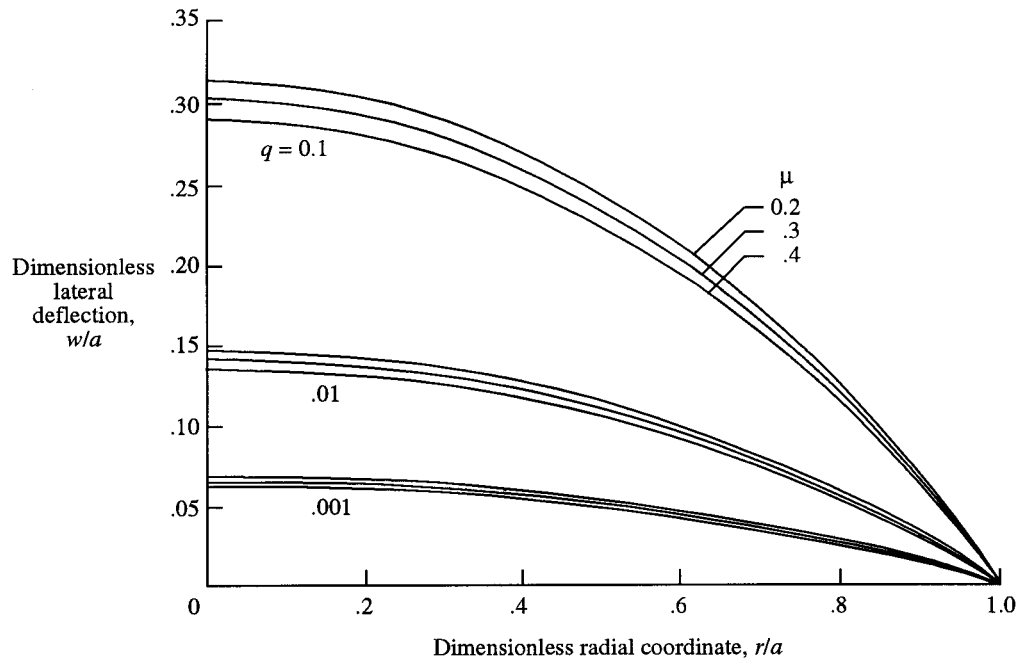


Figure 1. Lateral deflection due to uniform lateral loading for Hencky's problem.

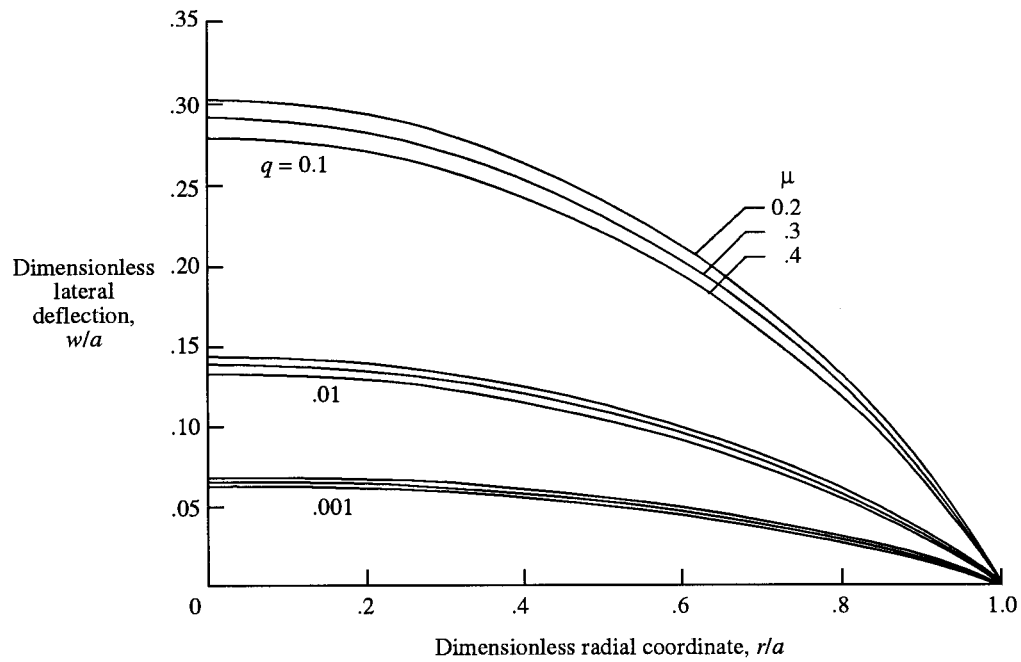


Figure 2. Lateral deflection due to uniform-pressure loading.

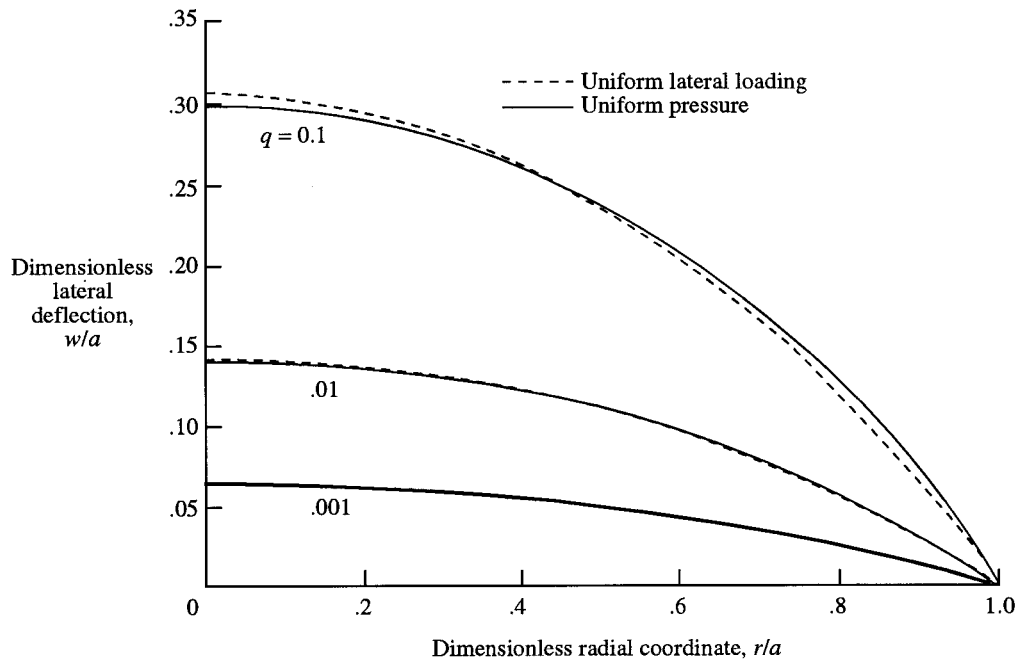


Figure 3. Lateral deflections for $\mu = 0.3$; $q = 0.001, 0.01$, and 0.1 .

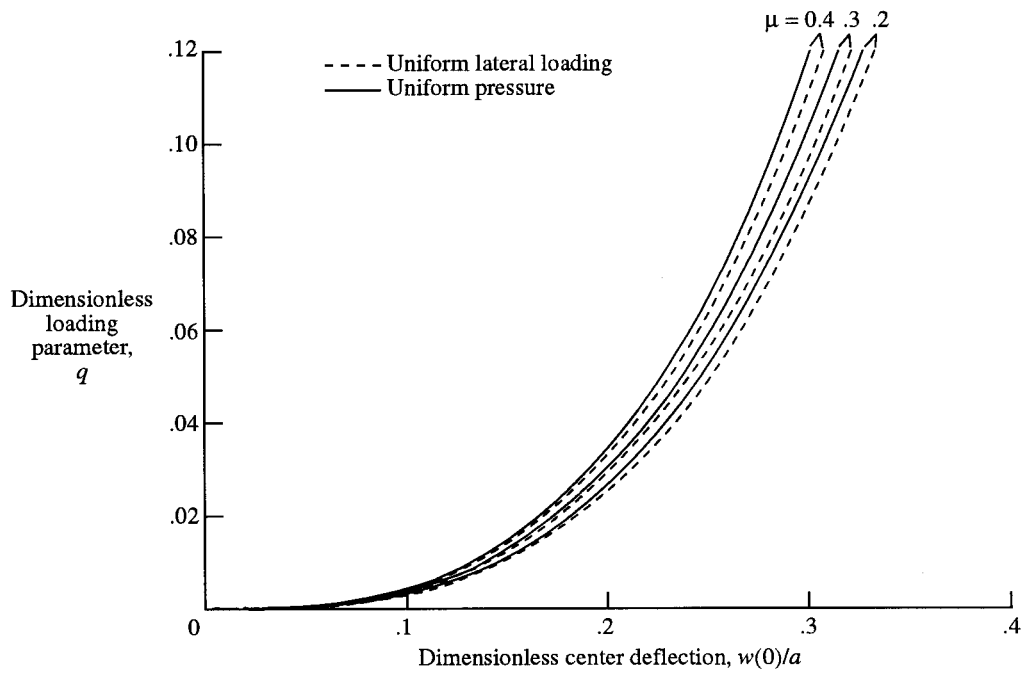


Figure 4. Dimensionless loading parameter as a function of center deflection.

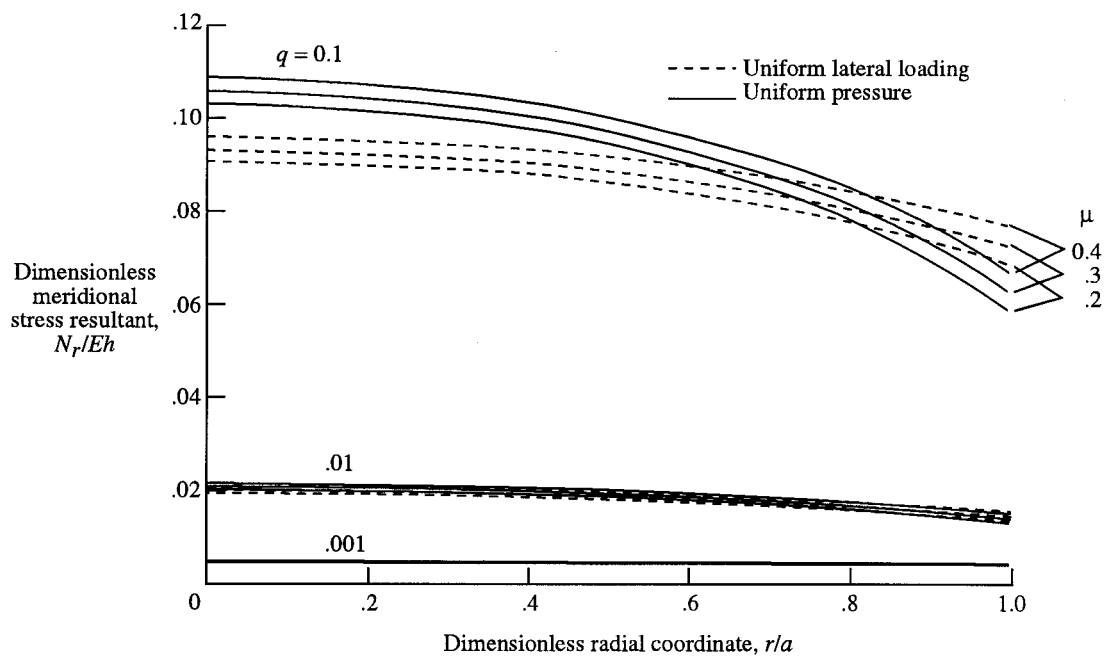


Figure 5. Meridional stress resultant distributions for $q = 0.001, 0.01$, and 0.1 .

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